GFDM Transceiver using Precoded Data and Low-complexity Multiplication in Time Domain

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Abstract—Future wireless communication systems are demanding a more flexible physical layer. GFDM is a block filtered multicarrier modulation scheme proposed to add multiple degrees of freedom and cover other waveforms in a single framework. In this paper, GFDM modulation and demodulation will be presented as a frequency-domain circular convolution, allowing for a reduction of the implementation complexity when MF, ZF and MMSE filters are employed in the inner and outer receiver operation. Also, precoding is introduced to further increase GFDM flexibility, addressing a wider set of applications.

Index Terms—precoding, low-complexity, multicarrier modulation, GFDM.

I. Introduction

Fifth generation (5G) networks are requiring a new level of flexibility on the physical layer (PHY). Although higher throughput keeps pushing the spectral efficiency to higher standards, new services will also demand very low latency, massive capacity for multiple connections and very low power consumption. The PHY must be flexible to address several different scenarios.

Generalized Frequency Division Multiplexing (GFDM) [1] is a recent waveform that can be engineered to address various use cases. GFDM arranges the data symbols in a time-frequency grid, consisting of M subsymbols and K subcarriers, and applies a circular prototype filter for each subcarrier. GFDM can be easily configured to cover other waveforms, such as Orthogonal Frequency Division Multiplexing (OFDM) and Single Carrier Frequency Domain Multiple Access (SC-FDMA) as corner cases. The subcarrier filtering can reduce the out-of-band (OOB) emissions, control peak to average power ratio (PAPR) and allow dynamic spectrum allocation. GFDM, with its block-based structure, can reuse several solutions developed for OFDM, for instance, the concept of a cyclic prefix (CP) to avoid inter-frame interference. Hence, frequency-domain equalization can be efficiently employed to combat the effects of multipath channels prior to the demodulation process. With these features, GFDM can address the requirements of 5G networks.

The main disadvantage of a flexible waveform is the complexity required for its implementation. Reducing the GFDM complexity is essential to bring its flexibility to 5G networks. The understanding of GFDM as a filtered multicarrier scheme

leads to a modem implementation based on time-domain circular convolution for each subcarrier, performed in the frequency-domain [1]. This paper follows the well known polyphase implementation of filter bank modulation [2]–[4] and applies it to GFDM.

As one main contribution, the investigation in this paper reveals that the Poisson summation formula [5] can be utilized in the demodulation process, considering a per-subsymbol circular convolution and decimation in the frequency-domain. This operation can be performed as an element-wise multiplication with subsequent simple M-fold accumulation in the time-domain. This simple re-orientation of the data symbol processing allows for a considerable reduction in complexity, which makes GFDM applicable in a wide range of scenarios.

This paper also contributes to reformulate the modulation and demodulation as matrix-vector operations based on $M(K)$-point discrete Fourier transforms (DFTs). As an outcome, it becomes evident that DFT can actually be considered as a precoding operation. Moreover, they can be replaced by other transformations as well. This observation facilitates arbitrary precoding of the GFDM data, which adds a new level of flexibility to the system.

It is worth mentioning that the proposed low complexity signal processing complements the work in [1] because the requirement for block alignment is loosened. It can be usable for supporting pipeline inner receiver implementation, particularly when building synchronization and channel estimation circuits for embedded training sequences. It can also be beneficial to develop future non-linear and recursive detection algorithms.

II. Classical GFDM Description and Low-Complexity Reformulation

Consider a wireless communication system that transmits data in a block-based structure that consists of K subcarriers and M subsymbols. Let $N = KM$ be the total number of data symbols in the block and $d_{k,m}$, $k = 0, \ldots, K - 1$, $m = 0, \ldots, M - 1$, denote the complex valued data symbol that is transmitted on the kth subcarrier and mth subsymbol. The classical description of GFDM signal generation [1] is given by

$$x[n] = \sum_{k=0}^{K-1} \left(g[n] \exp(j2\pi \tfrac{k}{K} n) \right) \circledast d_k[n] \qquad (1)$$

with $d_k[n] = \sum_{m=0}^{M-1} d_{k,m}\delta[n - mK], \qquad (2)$

where $n = 0, \ldots, N - 1$, $\circledast$ describes circular convolution carried out with period N and $g[n]$ denotes the impulse response of the transmit prototype filter.

Using the N-point DFT $\mathcal{F}_N$, (1) can be carried out as

$$x[n] = \mathcal{F}_N^{-1} \sum_{k=0}^{K-1} \{\underbrace{\mathcal{F}_N\{g[n]\exp(j2\pi\tfrac{k}{K}n)\}}_{G[\langle f-kM \rangle_N]} \underbrace{\mathcal{F}_N\{d_k[n]\}}_{\mathcal{F}_M\{d_{k,(\cdot)}\}|_f}\}, \quad (3)$$

where $\langle \bullet \rangle_N$ denotes the remainder modulo N and $G[f] = \mathcal{F}_N\{g[n]\}$. Due to periodicity of $\mathcal{F}_M$ with period M and $f = 0, \ldots, MK - 1$, $\mathcal{F}_M\{d_{k,(\cdot)}\}$ is concatenated K times. This leads to the low-complexity modulator implementation described in [1]. Now, expressing the convolution in (1) explicitly, the transmission equation can be reformulated as [4]

$$x[n] = \sum_{m=0}^{M-1} g[\langle n-mK \rangle_N] \underbrace{\sum_{k=0}^{K-1} d_{k,m} \exp(j2\pi\tfrac{k}{K}n)}_{K\mathcal{F}_K^{-1}\{d_{(\cdot),m}\}|_n}. \quad (4)$$

The sequence obtained from the Inverse Discrete Fourier Transform (IDFT) is concatenated M times in time-domain due to $n = 0, \ldots, MK - 1$. Such multiplication in the time-domain expresses the convolution of the subcarrier filter with the data in the frequency-domain for each subsymbol.

Assuming a flat and noiseless channel, the convolution of the received signal $y[n] = x[n]$ with a demodulation filter $\gamma[n]$, is given by

$$\hat{d}_{k,m} = \left(\gamma^*[\langle -n \rangle_N] \circledast y[n] \exp(-j2\pi\tfrac{k}{K}n)\right) |_{n=mK}. \quad (5)$$

With the help of DFT, (5) is written as

$$\hat{d}_{k,m} = \mathcal{F}_N^{-1}\{\underbrace{\mathcal{F}_N\{\gamma^*[\langle -n \rangle_N]\}}_{\Gamma^*[f]} \underbrace{\mathcal{F}_N\{y[n]\exp(-j2\pi\tfrac{k}{K}n)\}}_{Y[\langle f+kM \rangle_N]}\}|_{mK}, \quad (6)$$

where $\mathcal{F}_N\{\gamma[n]\} = \Gamma[f]$ and $\mathcal{F}_N\{y[n]\} = Y[f]$. For example, this leads to the low-complexity demodulator implementation described in [1] for the Matched filter (MF) $\gamma[n] = g[n]$, which was later extended to more general filter types in [6]. Now, directly expressing the convolution and sampling in (5) leads to the time-domain multiplication

$$\begin{aligned} \hat{d}_{k,m} &= \sum_{n=0}^{N-1} \gamma^*[\langle n-mK \rangle_N] y[n] \exp(-j2\pi\tfrac{kM}{KM}n) \\ &= \mathcal{F}_N\{\gamma^*[\langle n-mK \rangle_N] y[n]\}|_{kM}, \end{aligned} \quad (7)$$

where the N-point DFT is only evaluated at every Mth sample. According to the Poisson summation formula [5], this can be reduced to a $\frac{N}{M} = K$ point DFT

$$\mathcal{F}_N\{u[n]\}|_{kM} = \mathcal{F}_{\frac{N}{M}}\left\{\sum_{m=0}^{M-1} u[\langle n-mK \rangle_N]\right\}\Big|_k. \quad (8)$$

Eq. (7) describes a sampled short-time Fourier transform (STFT). This is clear since the GFDM demodulator is actually performing a Gabor transform [6], which can be understood as a sampled STFT. For this reason, (7) does not allow to employ
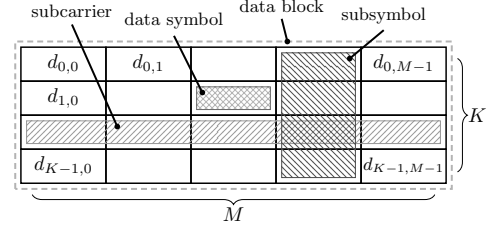

Fig. 1: Overview of block structure and terminology.

different prototype filters per subcarrier. This constraint is circumvented by performing equalization before demodulation or the presented scheme can be used to initially access the channel state information by demodulation of training data. Also, the proposed scheme facilitates the implementation of odd number of subsymbols [6] avoiding the use of M-point DFT and IDFT required in [1].

Notice that (7) holds when MF, zero-forcing (ZF) or minimum mean square error (MMSE) filters are used. ZF or MMSE filters can deliver good symbol error rate (SER) performance, while MF needs successive interference cancellation (SIC) to remove the SER floor caused by the self-interference. Hence, (7) shall be used to implement ZF and MMSE demodulators, while MF can be considered when the transmit filters are orthogonal or Offset Quadrature Amplitude Modulation (OQAM) is employed.

III. GFDM MATRIX MODEL

Arranging the data symbols $d_{k,m}$ in a two-dimensional structure leads to the data matrix $\mathbf{D} = \begin{pmatrix} \mathbf{d}_0 & \mathbf{d}_1 & \cdots & \mathbf{d}_{M-1} \end{pmatrix}$, where $\mathbf{d}_m = \begin{pmatrix} d_{0,m} & d_{1,m} & \ldots & d_{K-1,m} \end{pmatrix}^{\mathrm{T}}$ denotes the data symbols transmitted in the mth subsymbol.

According to the GFDM description in Section II, the rows and columns correspond to the time and frequency resources in Fig. 1, respectively. Hence, the kth row of $\mathbf{D}$ represents the data symbols transmitted in the kth subcarrier, $\mathbf{d}_k = \begin{pmatrix} d_{k,0} & d_{k,1} & \ldots & d_{k,M-1} \end{pmatrix}^{\mathrm{T}}$. The time-domain circular convolution of the modulation process can be expressed in the frequency-domain as [1]

$$\mathbf{x} = \frac{1}{N}\mathbf{W}_N^{\mathrm{H}} \sum_{k=0}^{K-1} \mathbf{P}^{(k)} \mathbf{G} \mathbf{R}^{(K,M)} \mathbf{W}_M \mathbf{d}_k. \quad (9)$$

For each subcarrier, the data vector is taken to the frequency-domain by the M-point DFT matrix $\mathbf{W}_M$. The corresponding time-domain up sampling operation is realized in the frequency-domain by duplicating the transformed data symbols vectors K times, using the repetition matrix $\mathbf{R}^{(K,M)} = \mathbf{1}_{K,1} \otimes \mathbf{I}_M$, where $\mathbf{I}_M$ is an M size identity matrix, $\mathbf{1}_{i,j}$ is a $i \times j$ matrix of ones and $\otimes$ is the Kronecker product. Each subcarrier is then filtered with $\mathbf{G} = \mathrm{diag}(\mathbf{W}_N\mathbf{g})$, where $\mathrm{diag}(\bullet)$ returns a matrix that contains the argument vector on its diagonal and zeros otherwise and $\mathbf{g}$ is the vector containing the transmit filter impulse response. An up-conversion of the kth subcarrier to its respective subcarrier frequency is performed by the shift matrix

$$\mathbf{P}^{(k)} = \mathbf{\Psi}\left(\mathbf{p}^{(k)}\right) \otimes \mathbf{I}_M, \quad (10)$$
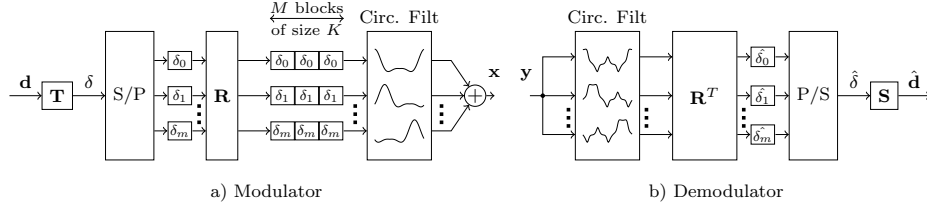
Fig. 2: Block diagram of low-complexity modulator and demodulator with precoding.

where $\boldsymbol{\Psi}(\bullet)$ returns the circulant matrix based on the input vector and $\mathbf{p}^{(k)}$ is a column vector where the kth element is 1 and all others are zero. The K subcarriers are summed and transformed back to the time-domain with $\mathbf{W}_N^H$ to compose the GFDM signal. On the demodulator side, the recovered data symbols for the kth subcarrier are given by

$$\hat{\mathbf{d}}_k = \frac{1}{M}\mathbf{W}_M^{\mathrm{H}}\left(\mathbf{R}^{(K,M)}\right)^{\mathrm{T}}\boldsymbol{\Gamma}\left(\mathbf{P}^{(k)}\right)^{\mathrm{T}}\mathbf{W}_N\mathbf{y}, \qquad (11)$$

where $\mathbf{y}$ is the equalized vector at the input of the demodulator, $\boldsymbol{\Gamma} = \mathrm{diag}(\mathbf{W}_N\boldsymbol{\gamma})$ with $\boldsymbol{\gamma}$ being the demodulation filter impulse response, e.g., MF, ZF or MMSE filters.

The modulation and demodulation processes can be simplified just by changing the processing order of the data symbols, as derived in (4). In this new matrix model, the GFDM vector is given by

$$\mathbf{x} = \sum_{m=0}^{M-1}\mathbf{P}^{(m)}\mathrm{diag}(\boldsymbol{g})\mathbf{R}^{(M,K)}\mathbf{W}_K^H\mathbf{d}_m. \qquad (12)$$

Observe that (12) is similar to a polyphase filter structure, but with the difference that cyclic time-shifts are used here and frequency-domain convolution is performed in time-domain as element-wise vector multiplication. With this approach, the first step is to obtain a time-domain version of $\mathbf{d}_m$ by multiplying it with an inverse DFT (IDFT) matrix $\mathbf{W}_K^{\mathrm{H}}$. M times upsampling in frequency-domain is performed in time by duplicating the transformed data symbols with a repetition matrix $\mathbf{R}^{(M,K)}$. Each subsymbol is then pulse-shaped with $\boldsymbol{g}$. $\mathbf{P}^{(m)}$ shifts the mth subsymbol to its respective position in time. The GFDM signal is obtained by summing all the pulse-shaped subsymbols, with no need of the N-IDFT domain conversion in (9).

The demodulation operations are also simplified by using the circular-convolution in frequency-domain, leading to

$$\hat{\mathbf{d}}_m = \mathbf{W}_K\left(\mathbf{R}^{(M,K)}\right)^{\mathrm{T}}\mathrm{diag}(\boldsymbol{\gamma})\left(\mathbf{P}^{(m)}\right)^{\mathrm{T}}\mathbf{y}. \qquad (13)$$

While the new approach describing GFDM modulation and demodulation as convolution in the frequency-domain reduces the implementation complexity, in the next section this principle will be expanded to a more general structure, where domain conversions will be understood as a precoding process.

IV. EXPANDING GFDM FEATURES WITH PRECODING

Eqs. (12) and (13) bring a new interpretation of the GFDM chain that further increases the flexibility of this waveform.

Let the transmit vector from (12) be redefined as

$$\mathbf{x} = \sum_{m=0}^{M-1}\mathbf{P}^{(m)}\mathrm{diag}(\boldsymbol{g})\mathbf{R}^{(M,K)}\boldsymbol{\delta}_m, \qquad (14)$$

where

$$\boldsymbol{\delta}_m = \mathbf{T}\mathbf{d}_m \qquad (15)$$

is the information coefficient vector to be transmitted and $\mathbf{T}$ is a transformation matrix. For classical GFDM, where a time-frequency grid is used to transmit the information, $\mathbf{T} = \mathbf{W}_K^H$. However, (15) can be seen as precoding the information vector by a generic matrix $\mathbf{T}$. Different matrices can be used to achieve different requirements. Hence, by transmitting precoded data, GFDM can be even more flexible to address requirements of future wireless networks.

On the receiver side, the recovered coefficients are given by

$$\hat{\boldsymbol{\delta}}_m = \left(\mathbf{R}^{(M,K)}\right)^{\mathrm{T}}\mathrm{diag}(\boldsymbol{\gamma})\left(\mathbf{P}^{(m)}\right)^{\mathrm{T}}\mathbf{y}. \qquad (16)$$

The estimated data symbols are obtained by reverting the precoding operation on (16), leading to

$$\hat{\mathbf{d}}_m = \mathbf{S}\hat{\boldsymbol{\delta}}_m. \qquad (17)$$

where $\mathbf{S}$ removes precoding. For conventional GFDM, $\mathbf{S} = \mathbf{W}_K$. A block diagram that described the proposed GFDM modem is presented in Fig. 2. The demodulation part can be implemented to run continuously, which can be used for tracking specific properties of the estimated information coefficient vector $\hat{\boldsymbol{\delta}}_m$ containing training data.

The precoding concept can be enlarged to embrace any transform taking into account the two dimensions of the GFDM data block $\mathbf{D}$, which is a bi-dimensional structure defined as a frequency-time grid spanning over several sub-carriers and subsymbols, as depicted in Fig. 1. Hence, in the default configuration, the columns of the matrix contain data defined in the frequency-domain, while the rows have data in the time-domain. The precoded data block $\boldsymbol{\Delta}$ can be generalized to

$$\underset{K\times M}{\boldsymbol{\Delta}} = \underset{K\times K}{\mathbf{T}_{\mathrm{c}}} \cdot \underset{K\times M}{\mathbf{D}} \cdot \underset{M\times M}{\mathbf{T}_{\mathrm{r}}}, \qquad (18)$$

where $\mathbf{D}$ is now a data block defined in an arbitrary domain, $\mathbf{T}_{\mathrm{c}}$ is a K by K precoding matrix applied to the columns and $\mathbf{T}_{\mathrm{r}}$ is a M by M precoding matrix applied to the rows of $\mathbf{D}$. For instance, choosing $\mathbf{T}_{\mathrm{c}}$ and $\mathbf{T}_{\mathrm{r}}$ to be Fourier matrices yields four domains for the GFDM data block, as given in Tab. I , where the domain frequency-time (FT) corresponds to standard GFDM. The choice of the data domain has impact on the robustness of the system regarding time and frequency
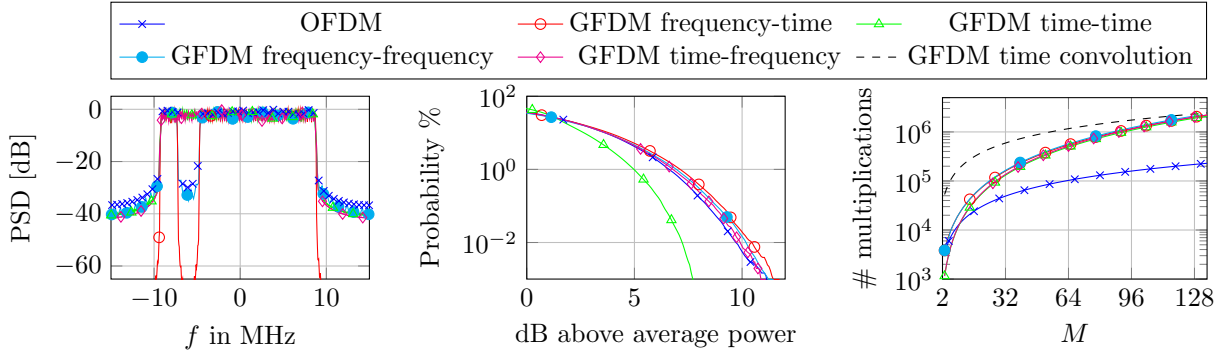
Fig. 3: OOB, PAPR and complexity analysis for precoded GFDM.
FT/FF: $M = 16$, $M_{\mathrm{on}} = 15$, $K = 128$, $K_{\mathrm{on}} = 75$, TT/TF: $K = 1$, $M = 1200$, OFDM: $N = 2048$, $N_{\mathrm{on}} = 1200$.

TABLE I: Precoding in time (T) and frequency (F) domains

Domain	Operation	Number of multiplications
FT	$\boldsymbol{\Delta} = \mathbf{W}_K^{\mathrm{H}} \mathbf{D} \mathbf{I}_M$	$MK \log_2(K)$
TT	$\boldsymbol{\Delta} = \mathbf{I}_K \mathbf{D} \mathbf{I}_M$	0
FF	$\boldsymbol{\Delta} = \mathbf{W}_K^{\mathrm{H}} \mathbf{D} \mathbf{W}_M^{\mathrm{H}}$	$MK \log_2(K) + KM \log_2(M)$
TF	$\boldsymbol{\Delta} = \mathbf{I}_K \mathbf{D} \mathbf{W}_M^{\mathrm{H}}$	$KM \log_2(M)$

selective fading. The concept is not limited to DFT and, in general, any meaningful transform can be applied to the rows and columns.

Taking the concept one step further, (18) can be formulated as

$$\mathrm{vec}(\boldsymbol{\Delta}) = \mathrm{vec}(\mathbf{T}_{\mathrm{c}} \mathbf{D} \mathbf{T}_{\mathrm{r}}). \tag{19}$$

With the help of the Kronecker product, the right part of (19) can be expressed as

$$\mathrm{vec}(\mathbf{T}_{\mathrm{c}} \mathbf{D} \mathbf{T}_{\mathrm{r}}) = \left(\mathbf{T}_{\mathrm{r}}^{\mathrm{T}} \otimes \mathbf{T}_{\mathrm{c}}\right) \mathrm{vec}(\mathbf{D}). \tag{20}$$

The resulting matrix

$$\mathcal{T} = \left(\mathbf{T}_{\mathrm{r}}^{\mathrm{T}} \otimes \mathbf{T}_{\mathrm{c}}\right) \tag{21}$$

can be seen as an even more general way of precoding that is not restricted to subcarriers or subsymbols but allows arbitrary coupling between any elements of $\mathbf{d} = \mathrm{vec}(\mathbf{D})$. The precoding is then applied leading to $\boldsymbol{\delta} = \mathcal{T}\mathbf{d}$.

V. OUT OF BAND, PAPR AND COMPLEXITY ANALYSIS

In particular, the FT mode allows for use of null subsymbols and subcarriers to support non-continuous bandwidth application, achieving a low OOB radiation as shown in Fig. 3. Although this is not possible with the time-time (TT) case, where the system essentially becomes a single-carrier system occupying the whole bandwidth, this precoding is especially beneficial for power-limited systems that cannot afford complex precoding operations with a low PAPR, as the PAPR can be significantly reduced. In contrast, frequency-frequency precoding (FF) results in a DFT-precoded GFDM system, with a spectrum similar to OFDM. In FF mode, data is defined directly in the frequency domain, according to (3), with one particular subsymbol only located on at most two frequency bins, assuming a transmit filter that is bandlimited to

two subcarriers. This property simplifies equalization, as only ICI remains, but on the other hand reduces frequency diversity in channels with strong frequency-selectivity. The last (TF) case also corresponds to the single carrier, but with symbols spread in time, creating potential for exploiting time-diversity.

The complexity of the precoding operation of the GFDM transceiver based on DFT algorithms are evaluated in Table I. Number of complex valued multiplications, a costly operation in implementations, is the chosen criterion for the analysis. The comparison is carried out under the assumption that $N = KM$ complex data symbols are transmitted. As the baseline for the comparison in Fig.3, MN multiplications from (16) are added to the precoding operations and the OFDM algorithm requires naturally the complexity of a DFT operation, with $N \log_2(N)$ multiplications. The GFDM modulator and demodulator proposed in [1] require the complexity of the M-point DFT algorithm with additional K times complex multiplications over repeated chunks of M complex samples, resulting from a DFT operation. Even when the frequency response of the demodulation filter is assumed to be sparse, e.g. when roll-off is small and a ZF filter span L can be smaller than the total number of subcarriers, the modem scheme presented in this paper is less complex than the solution proposed in [1].

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